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Falling body in the theories of gravitation

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Abstract. The motion of a test particle falling in the spherically symmetric field of another heavy body has been studied in a few of the theories of gravitation proposed recently. It is shown that while the predicted behaviour of the test particle in the new theories of gravitation agrees with that of the general theory of relativity (GTR) for small velocities and weak fields, the new theories predict certain results which are qualitatively different from those of GTR when the fields are strong and the velocities are high. For example, the falling body is shown to encounter *variable singular regions* and *repulsive fields* in the new theories of gravitation whereas in GTR the field is attractive everywhere outside the Schwarzschild sphere, for arbitrarily high velocities of the falling body.

1. Introduction

There have appeared in recent years, several theories of gravitation within the framework of flat space–time. These new theories of gravitation, more or less, are designed to meet the two requirements: (i) that they should reduce to the newtonian theory for weak fields and low velocities and (ii) 'explain' the three Einstein effects. Since, at least, a few of these theories, along with GTR, seem to explain the observed results of experimental gravitation fairly well, the need to find theoretical criteria to pick the 'correct' theory out of the mass of theories of gravitation has become very important. A fairly detailed analysis of the problem and the sorting of the theories is carried out by Will and Nordtvedt (1972a, b) using the PPN formalism. In this paper we compare a few of the new theories with GTR in the limit of strong fields and high velocities by studying the radial motion of a test particle in each of these theories. It is interesting that, although the orbital motion of a test particle in the centrally symmetric field of another heavy body has been studied in every theory of gravitation, the much simpler problem of the 'approaching motion' of the falling body does not appear to have received sufficient attention. It is the purpose of this paper to show that the falling body, which can probe any singularity in the neighbourhood of the attracting centre, can provide information which may be of some use in evaluating the theory.

As examples, we discuss the motion of the falling body according to the theories of Scott (1966), Rongved (1966) and Volkov (1971) and compare the results with those already obtained (Srinivasa Rao 1966, Zeldovich and Novikov 1971) according to GTR. Since all the theories reduce to the newtonian theory in the limit of low velocities and weak fields, they obviously yield the same motion for the test particle in this limit. However, when the velocity of the falling body is comparable with the velocity of light, the motion of the test particle is peculiar to each theory considered and the results are, in general, different.

2. Einstein's theory of gravitation

The motion of the falling body in the Schwarzschild field has been discussed elsewhere (Srinivasa Rao 1966, Zeldovich and Novikov 1971) and we quote here only the results for the sake of comparison. The velocity v of the falling particle is given by

$$\left(1-\frac{v^2}{c^2}\right)\left(1-\frac{S}{r_0}\right)=\left(1-\frac{u^2}{c^2}\right)\left(1-\frac{S}{r}\right) \tag{2.1}$$

where $S = 2GM/c^2$ is the Schwarzschild radius of the mass M, G is the universal constant of gravitation and $v = u$ at $r = r_0$. Here v refers to the physical velocity of the particle and is defined (Landau and Lifshitz 1951) as $(\mathrm{d}l/\mathrm{d}t)$ where $\mathrm{d}l$ and $\mathrm{d}t$ refer to elements of distance and physical time as measured by an observer at rest at the field point in question. The coordinate differentials $\mathrm{d}r$ and $\mathrm{d}x^0$ are related to $\mathrm{d}l$ and $\mathrm{d}t$ by

$$\mathrm{d}l^2=\mathrm{d}r^2\left(1-\frac{S}{r}\right)^{-1};\qquad \mathrm{d}t^2=\left(1-\frac{S}{r}\right)\left(\frac{\mathrm{d}x^0}{c}\right)^2.$$

In the limit $c\to\infty$, $r\to l$ and (2.1) reduces to the newtonian formula

$$v^2=u^2+2GM\left(\frac{1}{l}-\frac{1}{l_0}\right), \tag{2.2}$$

where

$$l=\int^r\left(1-\frac{S}{r}\right)^{-1/2}\mathrm{d}r.$$

From (2.1), it follows that v increases monotonically from the initial value u to c as $r\to S$. Thus we have a gravitational field which is *attractive* everywhere outside the singular sphere $r = S$.

3. The gravitokinetic field theory of Scott

According to this theory (Scott 1966), the equation of motion of a test particle of rest mass m_0 in the centrally symmetric field of another body of rest mass M is given by

$$\boldsymbol{F}=\frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v})=-\left(1-\frac{v^2}{c^2}\right)^{-2}GMm\frac{\boldsymbol{r}}{r^3}, \tag{3.1}$$

where $\boldsymbol{v}=(\mathrm{d}\boldsymbol{r}/\mathrm{d}t)$ is the velocity of the particle and $m = m_0(1-v^2/c^2)^{-1/2}$ is its momental mass. For radial fall, $\boldsymbol{v}$ is antiparallel to $\boldsymbol{r}$ and (3.1) can be written as

$$\left(1-\frac{v^2}{c^2}\right)\frac{\mathrm{d}v}{\mathrm{d}t}=\frac{GM}{r^2}, \tag{3.2}$$

where $v=|\boldsymbol{v}|$. Integrating (3.2) with the initial condition $v = u$ at $r = r_0$, we get

$$v^2-u^2-\frac{v^4-u^4}{2c^2}=2GM\left(\frac{1}{r}-\frac{1}{r_0}\right). \tag{3.3}$$

From (3.3) we see that v touches c at $r = R$ and becomes complex for $r < R$, where

$$R=2S\left[\frac{2S}{r_0}+\left(1-\frac{u^2}{c^2}\right)^2\right]^{-1}. \tag{3.4}$$

Thus, the region $r \leqslant R$ is *singular* wherein the motion of the test particle is devoid of any meaning. However, the radius of this singular sphere is *not* characteristic of the mass M like the Schwarzschild radius $S = 2GM/c^2$, as R is a function of u and r_0 and in fact for $u \sim c$, $R \sim r_0$ itself.

4. Rongved's theory of gravitation

In this theory (Rongved 1966), the velocity of the test particle moving in the radially symmetric field of another mass M is given by

$$v^2 = c^2 R(1-k^2 R^{1/2}), \tag{4.1}$$

where $R = (1-2S/r)$, $S = 2GM/c^2$ and k is a constant of integration to be fixed by initial conditions on v. (The constant k used here is the reciprocal of the constant A used by Rongved.) For a particle falling along a radial line with $v = u$ at $r = r_0$, we get

$$k^2 = \left(1-\frac{u^2}{c^2 R_0}\right) R_0^{-1/2}, \tag{4.2}$$

where $R_0 = (1-2S/r_0)$. We have obviously, $0 \leqslant R_0 \leqslant 1$ for $2S \leqslant r_0 \leqslant \infty$ and $R_0 < 0$ for $r_0 < 2S$. Thus, outside the sphere $r = 2S$, $k^2 \gtreqless 0$ according as $c^2 R_0 \gtreqless u^2$, where, for finite r_0, $c^2 R_0 < c^2$. Further, the mass m of the test particle given by (Rongved 1966)

$$m = \frac{m_0}{kR^{3/4}},$$

clearly shows that $k^2 \leqslant 0$ corresponds to unphysical situations. We thus have the strange result that, for finite r_0, the range $c^2 R_0 \leqslant u^2 < c^2$ is forbidden for u in contrast to GTR, where the only requirement on u is that $u < c$.

From (4.1), we see that the velocity v has a maximum at $r = r_m$, where

$$r_m = \left(\frac{18}{9-4k^{-4}}\right) S, \tag{4.3}$$

with the maximum velocity given by

$$v_m = \sqrt{\tfrac{4}{27}}\, ck^{-2}. \tag{4.4}$$

Since v vanishes at $r = 2S$ and has a maximum at $r = r_m$, the field must be *repulsive* in $2S < r < r_m$. However, this repulsive region of the field is *variable* as r_m depends on u and r_0. If u and r_0 are so chosen to make $k^2 > \frac{2}{3}$, then r_m is positive and by making $(k^2-\frac{2}{3})$ sufficiently small, the size of this repulsive region can be made as large as we please. For $k^2 = \frac{2}{3}$, r_m is infinite and $v_m = \sqrt{\tfrac{1}{3}}c$ which, obviously, is the largest of all v_m. If $k^2 < \frac{2}{3}$ (or if the initial velocity exceeds $\sqrt{\tfrac{1}{3}}c$), then there is no velocity maximum as $r_m < 0$ and the velocity of the particle continuously decreases from the initial velocity to zero as it approaches $r = 2S$, showing that the field, in this case, is everywhere repulsive.

It is appropriate to point out here, that repulsive regions apparently follow from GTR too (McVittie 1956), but then only in terms of the *coordinate velocity* and these difficulties disappear (Srinivasa Rao 1966, Zeldovich and Novikov 1971), if one considers the *physical velocity* which is the only one having physical significance.

5. The action at a distance theory of Volkov

The relevant equation in this theory (Volkov 1971) giving the velocity of a test particle of rest mass m_0 approaching another mass M at rest ($M \gg m_0$) is given by

$$\left(1-\frac{v^2}{c^2}\right)^{1/2} = \frac{m_0c^2+\psi}{E-\phi}, \tag{5.1}$$

where $\psi = -\beta GMm_0/r$, $\phi = -\alpha GMm_0/r$, α and β are numbers subject to the constraint $\alpha+\beta = 1$ and E is the total energy of the system which is a constant. In the case $\beta < 0$, after the constant E has been determined by the condition $v = u$ at $r = r_0$, it follows from (5.1) that

$$v \gtreqless u \qquad \text{as} \qquad u \lesseqgtr v_{\mathrm{m}}; \qquad v_{\mathrm{m}} = c\left(1-\frac{\beta^2}{\alpha^2}\right)^{1/2}, \tag{5.2}$$

and obviously v_{m} is real for $\beta = (1-\alpha) < 0$. The case $\beta > 0$ is of no interest as only a negative β; $\beta = -\frac{5}{2}$ yields a perihelic shift in the orbit of mercury (Volkov 1971) agreeing with the prediction of GTR. With this choice of β, we get

$$v_{\mathrm{m}} = \sqrt{\tfrac{24}{49}}c.$$

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